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The Brezis-Nirenberg problem on the sphere $\mathbf{S}^3$

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Abstract

The Brezis-Nirenberg problem in *Euclidian* space $\mathbf{R}^N$ $(N > 2)$ concerns the existence of positive solutions of the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u + u^{p_N} & \text{in} \quad \Omega, \\ \quad u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded domain and $p_N = (N+2)/(N-2)$ is the critical Sobolev exponent. Such solutions were found to exist if and only if $\lambda \in (\lambda^*, \lambda_1)$, where λ_1 is the principal eigenvalue of the Laplacian with Dirichlet boundary conditions and $\lambda^* = 0$ if $N \geq 4$ and $\lambda^* \in (0, \lambda_1)$ if $N = 3$.

In this lecture we review recent work on the same problem, but then on the sphere $\mathbf{S}^3$. In particular, we shall focus on *negative* values of λ. Whereas in Euclidian space there are no positive solutions in this range, on the sphere we find a surprising wealth of positive solutions of which the complexity increases as $\lambda \to -\infty$.